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Closedness and Hadamard well-posedness of the solution map for parametric vector equilibrium problems

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Abstract In this paper we introduced new definitions of vector topological pseudomonotonicity to study the parametric vector equilibrium problems. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters. The Hadamard well-posedness of parametric vector equilibrium problems is also analyzed by using the new definitions of vector topological pseudomonotonicity.

Keywords Parametric vector equilibrium problems · Vector topological pseudomonotonicity · Mosco convergence · Hadamard well-posedness

Mathematics Subject Classification (2000) 49N60 · 90C31

1 Introduction

Bogdan and Kolumbán [8] considered the parametric equilibrium problems governed by topological pseudomonotone maps depending on a parameter. They gave sufficient conditions for closedness of the solution map defined on the set of parameters. In this paper we generalize their results for parametric vector equilibrium problems by using two new definitions of vector topological pseudomonotonicity.

Let (X, σ) be a Hausdorff topological space and let P (the set of parameters) be another Hausdorff topological space. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, where C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$.

For a given $p \in P$ we consider the following parametric vector equilibrium problem, in short $(VEP)_p$:

Find $a_p \in D_p$ such that

$$f_p\left(a_p, b\right) \in C, \quad \forall b \in D_p,$$

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where D_p is a nonempty subset of X and $f_p : X \times X \to \mathcal{Z}$ is a given function.

If $C = \mathbb{R}_+$, then $(VEP)_p$ is called a parametric equilibrium problem.

It is well known that vector equilibrium problem contains as special cases several problems, namely, vector optimization problem, vector variational inequality problem, vector complementarity problem, vector Nash equilibrium problem etc.

Let us denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P \setminus \{p_0\}$. For existence of solutions see [3] and [16].

In the specialized literature, several properties of the solution map defined on the set of parameters for parametric weak vector equilibrium problem has been investigated. Anh and Khanh [2], Huang et al. [14] studied the stability for multivalued vector quasiequilibrium problems and parametric implicit vector equilibrium problems. Bianchi and Pini [7], Kimura and Yao [15] studied the sensitivity for parametric vector equilibria. Anh and Khanh [1], Gong [13] considered the continuity or Hölder continuity of the solution mappings for parametric vector equilibrium problems. The well-posedness for vector equilibrium problems is studied by Bianchi et al. [6].

The goal of this paper is to study the closedness of solution map for parametric vector equilibrium problems. The paper is organized as follows. The notions of the vector topological pseudomonotonicity are introduced in Sect. 2, where the notion of the Mosco convergence of the sets is also given. In Sect. 3 we prove our main result about the closedness of the solution map for parametric vector equilibrium problems. In the final section, we investigate the generalized Hadamard well-posedness of parametric vector equilibrium problems.

2 Preliminaries

In this section, we will introduce two new definitions of the vector topologically pseudomonotone bifunctions with values in $\mathcal{Z}$. First, the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [4], for a subset A of $\mathcal{Z}$ the suprema of A with respect to C is defined by

$$\operatorname{Sup} A = \left\{z \in \bar{A} : A \cap (z + \operatorname{Int} C) = \emptyset\right\}$$

and the infima of A with respect to C is defined by

$$\operatorname{Inf} A = \left\{z \in \bar{A} : A \cap (z - \operatorname{Int} C) = \emptyset\right\}.$$

For more detail see [11].

Let $(z_i)_{i\in I}$ be a net in $\mathcal{Z}$. Let $A_i = \left\{z_j : j \geq i\right\}$ for every i in the index set I. The limit inferior and the limit superior of the net (z_i), respectively, are given by

$$\operatorname{Liminf} z_i = \operatorname{Sup}\left(\bigcup_{i\in I} \operatorname{Inf} A_i\right) \quad \text{and} \quad \operatorname{Limsup} z_i = \operatorname{Inf}\left(\bigcup_{i\in I} \operatorname{Sup} A_i\right).$$

We are going to use a representative result.

Proposition 1 ([12], Theorem 2.1) *Let* $(z_i)_{i\in I}$ *be a net in* $\mathcal{Z}$ *convergent to* z *and let* $A_i = \{z_j : j \geq i\}$.

(i) If there is an index i_0 *such that, for every* $i \geq i_0$*, there exists* $j \geq i$ *with* $\operatorname{Inf} A_j \neq \emptyset$*, then* $z \in \operatorname{Liminf} z_i$.

(ii) If there is an index i_0 *such that, for every* $i \geq i_0$*, there exists* $j \geq i$ *with* $\operatorname{Sup} A_j \neq \emptyset$*, then* $z \in \operatorname{Limsup} z_i$.

We introduce two new definitions of vector topologically pseudomonotonicity which play a central role in our main results.

Definition 1 Let (X, σ) be a Hausdorff topological space, and let D be a nonempty subset of X. A function $f : D \times D \to \mathcal{Z}$ is called A-vector topologically pseudomonotone if for every $b \in D$, $v \in C^c$ and for each net $(a_i)_{i \in I}$ in D satisfying $a_i \overset{\sigma}{\to} a \in D$ and

$$\operatorname{Liminf} f(a_i, a) \subset C \tag{1}$$

then there is i_0 in the index set I such that

$$\overline{\left\{f\left(a_j, b\right) : j \geq i\right\}} \subset f(a, b) - v + C^c$$

for all $i \geq i_0$.

If $C^c = -\operatorname{Int} C$, then the A-vector topological pseudomonotonicity coincide with a slight generalization of vector topological pseudomonotonicity given by Chadli, Chiang and Huang in [10].

Definition 2 Let (X, σ) be a Hausdorff topological space, and let D be a nonempty subset of X. A function $f : D \times D \to \mathcal{Z}$ is called B-vector topologically pseudomonotone if for every $b \in D$, $v \in C^c$ and for each net $(a_i)_{i \in I}$ in D satisfying $a_i \overset{\sigma}{\to} a \in D$ and

$$\operatorname{Liminf} f(a_i, a) = \emptyset \quad \text{or} \quad \operatorname{Liminf} f(a_i, a) \cap C \neq \emptyset \tag{2}$$

then there is i_0 in the index set I such that

$$\overline{\left\{f\left(a_j, b\right) : j \geq i\right\}} \subset f(a, b) - v + C^c$$

for all $i \geq i_0$.

In Definitions 1 and 2, if $\mathcal{Z} = \mathbb{R}$, and if C is the set of all nonnegative real numbers, then we get back the well-known topological pseudomonotonicity introduced by Brézis [9].

Remark 1 Every B-vector topologically pseudomonotone function is A-vector topologically pseudomonotone.

These two types of vector topological pseudomonotonicities are not equivalent, as the following example shows.

Example 1 Let the real vector function $f : [0, 1] \times [0, 1] \to \mathbb{R}^2$ be defined by

$$f(a, b) = \begin{cases} (-a - 1, a - b) & \text{if} \quad a > 0 \\ (-1, b) & \text{if} \quad a = 0 \end{cases},$$

where the ordering cone C of $\mathbb{R}^2$ is the third quadrant, i.e.

$$C = \left\{(a, b) \in \mathbb{R}^2 : a \leq 0, b \leq 0\right\}.$$

The function f is A-vector topologically pseudomonotone, but is not B-vector topologically pseudomonotone.

For $a > 0$, f is continuous therefore it is vector topologically pseudomonotone in both senses. Let us study the case when $a = 0$.

If $a_n = 0$, for all $n \in \mathbb{N}$ one has the obvious inclusion

$$\overline{\{f(0, b) : n \geq 1\}} \subset f(0, b) - v + C^c, \quad \forall b \in [0, 1].$$

If there exists $k \in \mathbb{N}$ such that $a_k \neq 0$, then we obtain that

$$f(a_k, 0) \in \text{Liminf } f(a_n, 0). \tag{3}$$

Indeed, $f(a_k, 0)$ is an inferior point, because otherwise it has to exist $j > k$ such that

$$\left(-a_j - 1, a_j\right) \in (-a_k - 1, a_k) - \text{Int } C$$

meaning that

$$\begin{cases} -a_j - 1 > -a_k - 1 \\ a_j > a_k \end{cases}$$

which is a contradiction. In a similar way we can prove that $f(a_k, 0)$ is a superior point.

Since $f(a_k, 0) \in C^c$, from (3) it follows that

$$\text{Liminf } f(a_n, 0) \not\subset C.$$

Consequently, the function f is A-vector topologically pseudomonotone.

Let $a_n = 1/n$ for every $n \in \mathbb{N}$, then $(-1, 0) \in \text{Liminf } f(a_n, 0)$ due to Proposition 1. Therefore, the assumption

$$\text{Liminf } f(a_n, 0) \cap C \neq \emptyset$$

holds. Let $v = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $b = 1$ then do not exists $n_0 \in \mathbb{N}$ such that the condition

$$\overline{\{f(1/n, 1) : n \geq n_0\}} \subset f(0, 1) - v + C^c$$

applies. Indeed

$$\overline{\{(-1/n - 1, 1/n - 1,) : n \geq n_0\}} \not\subset (-1, 1) - \left(-\frac{1}{2}, \frac{1}{2}\right) + C^c \text{ for all } n_0 \in \mathbb{N}$$

which proves that the function f is not B-vector topologically pseudomonotone.

Let us consider σ and τ two topologies on X. Suppose that τ is stronger than σ on X.

For the parametric domains in $(VEP)_p$ we shall use a slight generalization of Mosco's convergence [17].

Definition 3 ([8], Definition 2.2.) Let D_p be subsets of X for all $p \in P$. The sets D_p converge to D_{p_0} in the Mosco sense ($D_p \overset{M}{\to} D_{p_0}$) as $p \to p_0$ if:

(i) for every subnet $(a_{p_i})_{i \in I}$ with $a_{p_i} \in D_{p_i}$, $p_i \to p_0$ and $a_{p_i} \overset{\sigma}{\to} a$ imply $a \in D_{p_0}$;
(ii) for every $a \in D_{p_0}$, there exist $a_p \in D_p$ such that $a_p \overset{\tau}{\to} a$ as $p \to p_0$.

3 Closedness of the solution map

This section is devoted to prove the closedness of the solution map for parametric vector equilibrium problems.

Theorem 1 *Let X be a Hausdorff topological space with σ and τ two topologies, where τ is stronger than σ. Let D_p be nonempty sets of X, and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P \setminus \{p_0\}$ and the following conditions hold:*

(i) $D_p \overset{M}{\to} D_{p_0}$;

(ii) *For each net of elements* $(p_i, a_{p_i}) \in GraphS$, *if* $p_i \to p_0$, $a_{p_i} \overset{\sigma}{\to} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \overset{\tau}{\to} b$ *then*

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right) \cap (-\operatorname{Int} C) \neq \emptyset.$$

(iii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is A-vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0, *i.e. for each net of elements* $(p_i, a_{p_i}) \in GraphS$, $p_i \to p_0$ *and* $a_{p_i} \overset{\sigma}{\to} a$ *imply* $(p_0, a) \in GraphS$.

Proof Let $(p_i, a_{p_i})_{i \in I}$ be a net of elements $(p_i, a_{p_i}) \in GraphS$, i.e.

$$f_{p_i}\left(a_{p_i}, b\right) \in C, \quad \forall b \in D_{p_i} \tag{4}$$

with $p_i \to p_0$ and $a_{p_i} \overset{\sigma}{\to} a$. By the Mosco convergence of the sets D_p we get $a \in D_{p_0}$. Moreover, there exists a net $\left(b_{p_i}\right)_{i \in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \overset{\tau}{\to} a$. From the assumption ii) we obtain that

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \cap (-\operatorname{Int} C) \neq \emptyset. \tag{5}$$

Since $-\operatorname{Int} C$ is an open cone, it follows that there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, such that

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right) \in -\operatorname{Int} C \text{ for all } i \in I. \tag{6}$$

By replacing b with b_{p_i} in (4) we get

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) \in C. \tag{7}$$

From (7) and (6) we obtain that

$$f_{p_0}\left(a_{p_i}, a\right) \in C, \quad \text{for all } i \in I.$$

Since C is closed, it follows

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \subset C.$$

Now, we can apply iii) and we obtain that for every $b \in D_{p_0}$, $v \in C^c$, there exists $j_1 \in I$ such that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, b\right) : i \geq j\right\}} \subset f_{p_0}(a, b) - v + C^c, \quad \forall j \geq j_1. \tag{8}$$

We have to prove that

$$f_{p_0}(a, b) \in C, \quad \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \overline{b}\right) \in C^c.$$

Let be $f_{p_0}\left(a, \overline{b}\right) = v$ where $v \in C^c$. From (8) we obtain that there exists $j_1 \in I$ such that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, \overline{b}\right) : i \geq j\right\}} \subset v - v + C^c = C^c, \quad \forall j \geq j_1. \tag{9}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p there exists $\left(\overline{b}_{p_i}\right)_{i \in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \overset{\tau}{\to} \overline{b}$. By using again the assumption ii), it follows that there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, for which

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i}, \overline{b}\right) \in -\operatorname{Int} C, \quad \text{for all } i \in I. \tag{10}$$

From (9) and (10) it follows that

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in C^c, \quad i \in I, \tag{11}$$

contradicting (4). Hence $(p_0, a) \in GraphS$. □

Remark 2 The Theorem 1 does not imply the Theorem 1 in [8] since the assumption $ii)$ can not be replaced by

(ii') For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$ then

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right) \cap (-C) \neq \emptyset.$$

The following example confirms this statement.

Example 2 Let $\sigma = \tau$ be the natural topology on $X = [0, 1]$. Let $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$, (∞ means $+\infty$ from real analysis), $D_p = [0, 1]$, $p \in P$. On P we consider the topology induced by the metric d given by $d(m, n) = |1/m - 1/n|$, $d(n, \infty) = d(\infty, n) = 1/n$, for $m, n \in \mathbb{N}$, and $d(\infty, \infty) = 0$. The ordering cone C is the third quadrant, i.e. $C = \left\{(a, b) \in \mathbb{R}^2 : a \leq 0, b \leq 0\right\}$.

Let the real vector functions $f_n : [0, 1] \times [0, 1] \to \mathbb{R}^2$ be given by $f_n(a, b) = (-2a - 1, a - b - 1/n)$, $n \in \mathbb{N}$ and the function f_∞ be defined as in example 1

$$f_\infty(a, b) = \begin{cases} (-a - 1, a - b) & \text{if} \quad a > 0 \\ (-1, b) & \text{if} \quad a = 0 \end{cases}.$$

The function f_∞ is A-vector topologically pseudomonotone, but is not B-vector topologically pseudomonotone. If $a_n = 1/n$ for every $n \in \mathbb{N}$ then the assumption $ii')$ in Remark 2 holds. Indeed, from Proposition 1 it follows that

$$(0, 0) \in \text{Liminf}\left(f_n\left(a_n, b_n\right) - f_\infty\left(a_n, b\right)\right)$$

when $b_n \to b$. We have $(n, 1/n) \in GraphS$ for each $n \in \mathbb{N}$, $S(\infty) = \emptyset$ therefore $0 \notin S(\infty)$. Hence S is not closed at ∞.

If we replace the assumption $ii)$ with $ii')$ we have to give a stronger condition to assumption $iii)$.

Theorem 2 *Let X be a Hausdorff topological space with σ and τ two topologies, where τ is stronger than σ. Let D_p be nonempty sets of X, and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P \setminus \{p_0\}$ and the following conditions hold:*

(i) $D_p \xrightarrow{M} D_{p_0}$;

(ii') *For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$ then*

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right) \cap (-C) \neq \emptyset.$$

(iii) *$f_{p_0} : X \times X \to \mathcal{Z}$ is B-vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0, i.e. for each net of elements $(p_i, a_{p_i}) \in GraphS$, $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$ imply $(p_0, a) \in GraphS$.

Proof The proof is given in the following three steps.

Step 1. Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in GraphS$, i.e.

$$f_{p_i}\left(a_{p_i}, b\right) \in C, \quad \forall b \in D_{p_i} \tag{12}$$

with $p_i \to p_0$ and $a_{p_i} \overset{\sigma}{\to} a$. By the Mosco convergence of the sets D_p we get $a \in D_{p_0}$. Moreover, there exists a net $\left(b_{p_i}\right)_{i\in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \overset{\tau}{\to} a$. From the assumption $ii')$ we obtain that

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \cap (-C) \neq \emptyset. \tag{13}$$

Step 2. We will prove that (13) and (12) imply

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) = \emptyset \ \text{ or } \ \text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \cap C \neq \emptyset.$$

For this we can distinguish two cases:

Case 1. $\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \cap (-\,\text{Int}\,C) \neq \emptyset.$
Since $-\,\text{Int}\,C$ is an open cone, it follows that there exists a subnet, denoted by the same indexes, such that

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right) \in -\,\text{Int}\,C, \quad \text{for all } i \in I. \tag{14}$$

By replacing b with b_{p_i} in (12) we get

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) \in C. \tag{15}$$

From (15) and (14) we obtain that

$$f_{p_0}\left(a_{p_i}, a\right) \in C, \quad \text{for all } i \in I.$$

Since C is closed, it follows

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \subset C \tag{16}$$

consequently

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) = \emptyset \ \text{ or } \ \text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \cap C \neq \emptyset.$$

Case 2. $\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \cap (-\,\text{Int}\,C) = \emptyset.$
We can suppose that

$$\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \in (-\,\text{Int}\,C)^c, \quad \forall i \in I \tag{17}$$

and

$$f_{p_0}\left(a_{p_i}, a\right) \in C^c, \quad \forall i \in I \tag{18}$$

otherwise we get back the first case.
Since $\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \cap (-\,\text{Int}\,C) = \emptyset$, from (13) and (17) it follows that, there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, for which

$$\begin{aligned}&\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right)_{i\in I} \text{ converges}\\ &\text{to the boundary of cone } -C.\end{aligned} \tag{19}$$

Indeed, otherwise it must exist $i_0 \in I$ such that

$$\overline{\left\{f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right) : i \geq i_0\right\}} \subset (-C)^c$$

then from the definition of the limit inferior, we obtain that

$$\text{Liminf}\left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, a\right)\right) \subset (-C)^c,$$

which is in contradiction with assumption ii').
From (18) and (19) we obtain that there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, such that

$$\left(f_{p_0}\left(a_{p_i}, a\right)\right)_{i\in I} \text{ converges to an element in the boundary of the cone } C. \tag{20}$$

To prove this statement, let us suppose the contrary, that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, a\right) : i \in I\right\}} \subset C^c. \tag{21}$$

Then from (19) we obtain that

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) \text{ converges to an element in } C^c.$$

Since C^c is an open cone, it follows that there exists $i_1 \in I$ such that

$$f_{p_i}\left(a_{p_i}, b_{p_i}\right) \in C^c \text{ for all} ai \geq i_1,$$

contradicting (12).
By applying the Proposition 1 for the subnet in (20) we obtain that

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \cap (\partial C) \neq \emptyset,$$

or there exists $i_2 \in I$ such that

$$\text{Inf}\left\{f_{p_0}\left(a_{p_i}, a\right) : i \geq i_2\right\} = \emptyset.$$

This implies that

$$\text{Liminf } f_{p_0}\left(a_{p_i}, a\right) \cap C \neq \emptyset \text{ or } \text{Liminf } f_{p_0}\left(a_{p_i}, a\right) = \emptyset.$$

So, in both cases, we can apply iii) and we obtain that for every $b \in D_{p_0}$ and $v \in C^c$, there exists $j_0 \in I$ such that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, b\right) : i \geq j\right\}} \subset f_{p_0}(a, b) - v + C^c, \quad \forall j \geq j_0. \tag{22}$$

Step 3. We have to prove that

$$f_{p_0}(a, b) \in C, \quad \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \overline{b}\right) \in C^c.$$

Let be $f_{p_0}\left(a, \overline{b}\right) = v$ where $v \in C^c$. From (22) we obtain that there exists $j_0 \in I$ such that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, \overline{b}\right) : i \geq j\right\}} \subset v - v + C^c = C^c, \quad \forall j \geq j_0. \tag{23}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p, we have that there exists $\left(\overline{b}_{p_i}\right)_{i\in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \xrightarrow{\tau} \overline{b}$. By using again the assumption ii'), it follows that one of the next cases, corresponding to (14) and (19), respectively, hold:

there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, such that

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i}, \overline{b}\right) \in -\operatorname{Int} C, \quad \forall i \in I \tag{24}$$

or there exists a subnet $\left(a_{p_i}\right)$, denoted by the same indexes, for which

$$\begin{aligned}&\left(f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i}, \overline{b}\right)\right)_{i \in I} \text{ converges}\\ &\text{to the boundary of cone } -C.\end{aligned} \tag{25}$$

From (23), (24) and (25) it follows that there exists $j_1 \in I$ such that

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in C^c, \quad i \geq j_1 \geq j_0, \tag{26}$$

but on other side $(p_i, a_{p_i}) \in GraphS$, and

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in C,$$

which is a contradiction. Hence $(p_0, a) \in GraphS$.

□

Let $\mathcal{Z} = \mathbb{R}$ and $C = [0, +\infty)$ in Theorem 1, then we obtain the following result.

Corollary 1 ([8], Theorem 1) *Let X be a Hausdorff topological space with σ and τ topologies on X, where τ is stronger than σ. Let D_p be nonempty sets of X, $p \in P$, and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$, for each $p \in P$, and the following apply:*

(i) $D_p \xrightarrow{M} D_{p_0}$;

(ii) $f_p : X \times X \to \mathbb{R}$ *satisfies the following condition at* p_0 :

(C) *For each net of elements $(p_i, a_{p_i}) \in Graph\ S$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$, then*

$$\liminf \left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \leq 0.$$

(iii) $f_{p_0} : X \times X \to \mathbb{R}$ *is topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0, i.e. for each net of elements $(p_i, a_{p_i}) \in Graph\ S$, $p_i \to p_0$ and $a_{p_i} \to a$ imply $(p_0, a) \in Graph\ S$.

4 Hadamard well-posedness

Let us recall some classical definitions from set-valued analysis. Let X, Y be topological spaces. The map $T : X \to 2^Y$ is said to be *upper semi-continuous* at $u_0 \in domT := \{u \in X | T(u) \neq \emptyset\}$ if for each neighborhood V of $T(u_0)$, there exists a neighborhood U of u_0 such that $T(U) \subset V$. The map T is considered to be *closed* at $u \in domT$ if for each net $(u_i)_{i \in I}$ in $domT$, $u_i \to u$ and each net $(y_i)_{i \in I}$, $y_i \in T(u_i)$, with $y_i \to y$ one has $y \in T(u)$. The map T is said to be *closed* if its graph $GraphT = \{(u, y) \in X \times Y | y \in T(u)\}$ is closed, namely if $(u_i, y_i) \in GraphT$, $(u_i, y_i) \to (u, y)$ then $(u, y) \in GraphT$.

Closedness and upper semi-continuity of a multifunction are closely related.

Proposition 2 ([5] Proposition 1.4.8, 1.4.9) *Let X, Y be Hausdorff topological spaces.*

(i) *If $T : X \to 2^Y$ has closed values and is upper semi-continuous then T is closed;*

(ii) *If Y is compact and T is closed at $x \in X$ then T is upper semi-continuous at $x \in X$.*

Now we recall the notion of generalized Hadamard well-posedness.

Definition 4 The problem $(VEP)_p$ is said to be Hadamard well-posed (briefly H-wp) at $p_0 \in P$ if $S(p_0) = \{a_{p_0}\}$ and for any $a_p \in S(p)$ one has $a_p \overset{\sigma}{\to} a_{p_0}$, as $p \to p_0$. The problem $(VEP)_p$ is said to be generalized Hadamard well-posed (briefly gH-wp) at $p_0 \in P$ if $S(p_0) \neq \emptyset$ and for any $a_p \in S(p)$, if $p \to p_0$, (a_p) must have a subsequence σ-converging to an element of $S(p_0)$.

With the help of the next result we are able to establish the relationship between upper semi-continuity and Hadamard well-posedness.

Proposition 3 ([18] Theorem 2.2) *Let X and Y be Hausdorff topological spaces and $T : X \to 2^Y$ be a set valued map. If T is upper semi-continuous at $x \in X$ and $T(x)$ is compact, then T is gH-wp at x. If more, $T(x) = \{y^*\}$, then T is H-wp at x.*

In the following we prove that the solution map of $(VEP)_p$ has closed value at p_0.

Proposition 4 *If D_{p_0} is closed with respect to the σ topology and $f_{p_0} : X \times X \to \mathcal{Z}$ is A-vector topologically pseudomonotone, then $S(p_0)$ is closed with respect to the σ topology.*

Proof Let $S(p_0) \neq \emptyset$ and $a_i \in S(p_0)$, with $a_i \overset{\sigma}{\to} a$. Since D_{p_0} is closed with respect to the σ topology, we have that $a \in D_{p_0}$. From $a_i \in S(p_0)$ it follows that

$$f_{p_0}(a_i, a) \in C, \quad \forall i \in I.$$

Since C is closed, we get

$$\text{Liminf}\ f_{p_0}(a_i, a) \subset C.$$

By using the A-vector topological pseudomonotonicity we obtain that for every $b \in D_{p_0}$ and $v \in C^c$ there is j_0 in the index set I such that

$$\overline{\{f(a_i, b) : i \geq j\}} \subset f(a, b) - v + C^c, \quad \text{for all } j \geq j_0. \tag{27}$$

We have to prove that $a \in S(p_0)$, i.e.

$$f_{p_0}(a, b) \in C, \quad \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \overline{b}\right) \in C^c.$$

Let $f_{p_0}\left(a, \overline{b}\right) = v$ where $v \in C^c$. From (27) we obtain that

$$\overline{\left\{f_{p_0}\left(a_i, \overline{b}\right) : i \geq j\right\}} \subset v - v + C^c = C^c, \quad \forall j \geq j_0$$

which is a contradiction to $a_i \in S(p_0)$. Thus $a \in S(p_0)$. □

Now we can formulate the following results.

Corollary 2 *Let (X, σ) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let D_p be nonempty sets of X, and D_{p_0} be a closed subset of X. If the hypotheses of Theorem 1 are satisfied, then $(VEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(VEP)_p$ is Hadamard well-posed at p_0.*

Proof From Theorem 1 we obtain that the solution map S is closed at p_0. By using Proposition 2 ii) it follows that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Proposition 4, hence it is compact. The conclusion follows from Proposition 3. □

From Remark 1 and Corollary 2, we obtain:

Corollary 3 *Let* (X, σ) *be a compact Hausdorff topological space and* P *be a Hausdorff topological space. Let* D_p *be nonempty sets of* X*, and* D_{p_0} *be a closed subset of* X*. If the hypotheses of Theorem 2 are satisfied, then (VEP)*$_p$ *is generalized Hadamard well-posed at* p_0*. Furthermore, if* $S(p_0) = \{a_{p_0}\}$ *(a singleton), then (VEP)*$_p$ *is Hadamard well-posed at* p_0.

References

1. Anh, L.Q., Khanh, P.Q.: On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems. J. Math. Anal. Appl. **321**, 308–315 (2006)
2. Anh, L.Q., Khanh, P.Q.: On the stability of the solution sets of general multivalued vector quasiequilibrium problems. J. Optim. Theory Appl. **135**, 271–284 (2007)
3. Ansari, Q.H., Flores-Bazán, F.: Recession methods for generalized vector equilibrium problems. J. Math. Anal. Appl. **321**, 132–146 (2006)
4. Ansari, Q.H., Yang, X.C., Yao, J.C.: Existence and duality of implicit vector variational problems. Numer. Funct. Anal. Optim. **22**, 815–829 (2001)
5. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston, Massachusetts (1990)
6. Bianchi, M., Kassay, G., Pini R.: Well-posedness for vector equilibrium problems. Math. Meth. Oper. Res. **70**, 171–182 (2009). doi:10.1007/s0018600802394
7. Bianchi, M., Pini, R.: Sensitivity for parametric vector equilibria. Optimization **55**, 221–230 (2006)
8. Bogdan, M., Kolumbán, J.: Some regularities for parametric equilibrium problems. J. Glob. Optim. **44**, 481–492 (2009). doi:10.1007/s1089800893453
9. Brézis, H.: Equations et inequations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier (Grenoble) **18**, 115–175 (1968)
10. Chadli, O., Chiang, Y., Huang, S.: Topological pseudomonotonicity and vector equilibrium problems. J. Math. Anal. Appl. **270**, 435–450 (2002)
11. Chiang, Y.: Vector superior and inferior. Taiwan. J. Math. **8**, 477–487 (2004)
12. Chiang, Y., Yao, J.C.: Vector variational inequalities and the $(S)_+$ condition. J. Optim. Theory Appl. **123**, 271–290 (2004)
13. Gong, X.H.: Continuity of the solution set to parametric weak vector equilibrium problems. J. Optim. Theory Appl. **139**, 35–46 (2008)
14. Huang, N.J., Li, J., Thompson, H.B.: Stability for parametric implicit vector equilibrium problems. Math. Comput. Model. **43**, 1267–1274 (2006)
15. Kimura, K., Yao, J.C.: Sensitivity analysis of solution mappings of parametric vector quasi-equilibrium problems. J. Glob. Optim. **41**, 187–202 (2008)
16. Lin, L.J.: Existence Results for primal and dual generalized vector equilibrium problems with applications to generalized semi-infinite programming. J. Glob. Optim. **33**, 579–595 (2005)
17. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
18. Yang, H., Yu, J.: Unified approaches to well-posed with some applications. J. Glob. Optim. **31**, 371–381 (2005)